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Portfolio Insurance Contracts Linked to Hedge Funds

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ABSTRACT

We stress the importance of modeling the evolution of hedge funds NAV following a jump-diffusion process in order to properly assess a contract of portfolio insurance managed following the cushion method (CPPI). This choice is explained partly by the statistical properties of hedge fund returns, and also by the impact on the price of such contracts through the increase in the price of the Gap option embedded in the contract. We show through the use of non-parametric statistical methods (Barndorff-Nielsen and Shephard (2004), and Bollerslev et al. (2008)) that returns of hedge funds have a level of jump activity similar to that present in the equity indices. Then we show the existence of a common factor between hedge funds and equity indices corresponding to a systemic jump. Finally, we present numerical results in order to understand the impact of the various parameters of the jump-diffusion process on the CPPI valuation.

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1. Introduction

Portfolio Insurance refers to portfolio management techniques that allow the investor to guarantee a fixed percentage of the initial investment while participating in case of an upside trend. There exists two main strategies of portfolio insurance: Option Based Portfolio Insurance (OBPI), introduced in Leland and Rubinstein (1976). It consists in buying simultaneously the underlying asset chosen by the investor (index, stock, fund...), and a put on this underlying. Or equivalently, it consists in buying a call on the underlying plus a zero-coupon bond (which is obtained by the Put-Call Parity). The second strategy named Constant Proportion Portfolio Strategy (CPPI), has been developped by Perold (1986) and Black and Jones (1987). This method is based on particular rules of reallocation between the riskless asset, and the risky one. The investor chooses the lowest acceptable value of the portfolio, and the related acceptable exposure to the risky asset through the choice of the *cushion* and of the *multiplier*.

The recent growth of the hedge funds industry, and their emergence as an alternative asset class attracted the interest of investors for this type of investment by holding shares of hedge funds, or more recently through strategies of portfolio insurance linked to hedge funds. The objective of the investor in portfolio insurance strategies is to limit the losses, and to take part to the rises through a chosen participation.

In the case of hedge funds, the most appropriate strategy is CPPI, because OBPI would require buying some quantity of puts on the hedge fund, which could be very expensive and even impossible in many cases. The recourse to the strategy of CPPI when the underlying is a hedge fund rises however, various problems. One concern with the pricing of a CPPI strategy is the *gap risk* inherent to it which materializes when, between two respective reallocation dates, the wealth drops below the current floor level. In this case, the CPPI manager is no longer able to provide all the guaranteed capital at maturity. This issue is especially important when CPPI strategies are written on hedge funds because of the statistical evidence that their returns are not gaussian[1]. Several studies have recently presented empirical evidence on the presence of jumps in financial asset prices using non-parametric tests with high-frequency data, see e.g., Barndorff-Nielsen and Shephard (2004), Andersen et al. (2003), Ait-Sahalia and Jacod (2009), and Jacod and Todorov (2009). In the sequel we follow the methodologies proposed Barndorff-Nielsen and Shephard (2004), and Bollerslev et al. (2008) in order to test for the existence of jump activity in hedge fund indices, by examining monthly prices and estimated

[1] See Ezzili (2008) for more details about this issue.

daily ones. Using the same non-parametric methodologies, we give some evidence of the existence of a systematic jump component common with that of stock indices. In order to take into account negative jumps of hedge funds, we model them as following a jump-diffusion model. CPPI strategies in presence of jumps in the underlying were also considered in Prigent and Tahar (2005). Cont and Tankov (2007) considered recently various risk measures for the loss function with that respect and illustrated its results in the case of Kou's jump-diffusion model with parameters estimated from daily returns of two stocks, while we estimate the parameters of Merton's jump-diffusion from daily returns of HFR hedge fund indices.

This paper is organized as follows. Section 1 presents the general setting and a jump-diffusion model for the hedge fund NAV process. In Section 2, we investigate the existence of jump activity in hedge fund prices, and whether there is a common jump factor between hedge funds indices and equity ones, and we find some evidence for this. In Section 3, we derive the value of the CPPI strategy. We evaluate the *Gap Option* in Section 4, and Section 5 describes numerical results.

2. Modeling the Hedge Fund NAV

Goetzmann et al. (2003) and Atlan (2007) modeled the hedge fund's NAV (which is the value of the fund for one dollar invested) as a lognormal diffusion process according to the following linear stochastic differential equation:

$$dV(t) = V(t)(\mu dt + \sigma dZ(t)) \tag{1}$$

where $(Z_t)_{\{t\in[0,T]\}}$ is a one-dimensional Brownian-motion under P, the historical probability measure.

Atlan (2007) adopted the view that there exists an equivalent measure Q under which the risk-neutral dynamic followed by the NAV is:

$$dV(t) = V(t)((r + \alpha - c - f(V(t)))dt + \sigma dZ^{\mathsf{Q}}(t)) \tag{2}$$

where α denotes the premium return on the fund's assets defined by the classical CAPM relationship:

$$\mu - r = \alpha + \beta(\mu_m - r)$$

r_m being the expected return on the market's portfolio.

c is a predetermined constante denoting the management fees, which are proportional to the

assets under management, while $f(V(t)) = \mu\Phi 1_{\{V(t)>He^{rt}\}}$ represents the incentive fees, contingent to the current NAV's value attaining the high water mark level[2] (H being the initial high water mark level, and Φ denotes the incentive fees rate). The term $c + f(V(t))$ can be interpreted as the continuous dividend-yield of the hedge fund.

2.1 Jump-Diffusion Process for Hedge Funds

We adopt the view that the hedge fund's NAV reflects better the reality if we model it as a jump-diffusion process, which make possible to take into account the negative jumps characterizing hedge funds (econometric evidence of the presence of jumps is given in Section 2). Contrary to Atlan (2007), we assume that the observed NAV is net of fees, but we include a term r_f corresponding to the retrocession fees negotiated by the CPPI manufacturer as an important actor in the hedge fund's life (in many cases the issuance of the CPPI is intended to finance the fund's growth). By usual arguments and considering the hedge fund's NAV reinvested at the retrocession fees rate as a martingale (see Schroder (1999)), the NAV's dynamic satisfy the following SDE under the risk-neutral probability Q:

$$dV(t) = V(t^-)[(r - r_f)dt + \sigma dZ^Q(t) + \gamma(dN(t) - \lambda dt)] \tag{3}$$

We have chosen a Merton-type jump-diffusion process, which is a combination of a Brownian motion with a drift and a compound Poisson process, and where the jumps have a Gaussian distribution $N(b,\delta)$. $Z^Q(t)$ is a one-dimensional Brownian process, $N(t)$ is a univariate compound Poisson process of constant intensity λ, and γ denotes the jump amplitude. Moreover, it is assumed that the Brownian process and the compound Poisson process are independent.

3. Estimation of Jump-Diffusion Processes

The jump-diffusion parameters can be estimated historically or calibrated to market data. In the latter case, the calibrated jump-diffusion process better reflects the agents' anticipations of future jumps, which is more consistent with the hedging. However, in the case of illiquid underlyings such as hedge funds, out-of-the-money puts are rarely traded, hence calibrating to market data does not make sense. We propose to disentangle the specific risk present in the historical data of a single hedge fund (or a hedge fund indice) from the systematic risk (*hedgeable* through options on main market indices). The jump term can be decomposed into

[2] See Ezzili (2008) for more details about the mechanism of fee schemes in the hedge fund industry.

two components: one specific that is estimated empirically, and another one that is systematic. We give in the sequel some elements of evidence to the assumption that the observed jumps in the hedge funds are partially systematic (and correlated with observed negative jumps in stock markets).

3.1 Empirical Estimation of Jump-Diffusion Processes

In order to estimate the Merton's parameters vector $\theta = (\mu, \sigma, \lambda, b, \delta)$ from historical data, we use the empirical characteristic function estimation [3] which aims at minimizing the integral over u of a weighted distance between the empirical characteristic function and the theoretical characteristic function

$$\int_{\infty}^{\infty} | c_\theta(u) - c(u,n) |^2 \; g(u) du$$

where

$$c(u,n) = \frac{1}{N} \sum_{k=1}^{N} e^{iuX_i}$$

is the empirical characteristic function,

$$c_\theta(u) = exp\{iu\mu T - \frac{1}{2} u^2 \sigma^2 T + \lambda T (e^{iub - u^2 \delta^2 /2} - 1)\}$$

is the characteristic function of the Merton's model and

$$g(u) = exp(-u^2)$$

is the weight function used. Notice that this corresponds to a standard choice (see for example Yu (2004)). Cont and Tankov (2007) use a weight function $g(u) = \dfrac{1}{1+\alpha u^2}$ with $\alpha = 0.01$ in order to attribute more weights to the tails of the distribution for the estimation of Kou's model[4].

The optimal weight obtained by Feuerverger and McDunnough (1981) using the Parsaval identity is given by

[3] See Singleton (2001), Yu (2004) and Rockinger and Semenova (2005) for details on this method. We also present some details in Appendix.
[4] We have also tried estimation using this specification of the weight but it didn't seem to change the estimation's results.

$$g^*(u) = \frac{1}{2\pi}\int exp(-irx)\frac{\partial \log f_\theta(x)}{\partial\theta}dx$$

The weight is optimal in the sense that the obtained estimator attains the maximum likelihood efficiency. However, when the likelihood function has no closed form expression (no closed form for the density function), which represents the majority of interesting cases, the optimal weight is unknown.

We estimate[5] the parameters of Merton's jump-diffusion model from daily returns of 12 HFR hedge fund indices. The obtained parameters are reported in Table 5.1. We observe that the estimated σ is systematically equal to zero, while the mean jump b is always negative, which suggests that hedge funds indices seem to be only driven by a jump activity. However the versatility of the obtained parameters with the initial conditions choice encourages us to remain more cautious on the interpretation.

Table 1
Empirical estimation of Merton's model parameters from HFR daily time series

	Observations	μ	σ	λ	b	γ
HFR Equal Weighted Strat.	1109	0.0002	0.0000	0.2874	-0.0940	0.0003
HFR US Absolute Return	793	0.0001951	0.0100	0.9574	-0.0508	0.0054
HFR US Convertible Arb.	1109	0.0001	0.0000	0.1037	-0.00105	0.0000
HFR US Distressed S.	1109	0.0003	0.0000	0.1859	-0.0503	0.0001
HFR US Equity Market N.	1109	0.0000523	0.0100	0.0100	-0.5743	0.0224
HFR US Equity Hedge	1109	0.0003	0.0000	0.172	-0.0029	0.0000
HFR US Event Driven	1109	0.0004	0.0000	0.1363	-0.0039	0.0003
HFR US Global HF	1109	0.0003	0.0000	0.0849	-0.0007	0.0000
HFR US Macro Index	1109	0.0001	0.0000	0.1625	-0.0044	0.0012
HFR US Market Directional	793	0.0002	0.0000	0.033	-0.0002	0.0000
HFR US Merger Arb.	1109	0.0002	0.0000	0.1468	-0.0043	0.0000
HFR US Relative Value Arb.	1109	0.0002	0.0000	0.0524	-0.0018	0.0001

In the next sub-section, we examine whether there exists common jump factors between hedge funds and stock indices.

3.2 Are Jumps Common Factor?

It is commonly admitted that hedge funds are good "diversificator" (see Amin and Kat (2002) for example) due to their low correlations with current market indices. However, we observe that during crisis periods the returns of hedge funds become more correlated with those of the other asset classes. The most significant example is the Russian crisis in 1998 (see Jorion

[5] See details about the estimation algorithm in Appendix.

(1999)) which caused the bankruptcy of LTCM and by a spillover effect had a huge impact on the hedge funds industry and on all the stock markets.

3.2.1 Sample

We consider monthly data of HFR indices (Convertible Arbitrage Index, Distressed Securities Index, Statistical Arbitrage Index, Equity Hedge Index, Equity Market Neutral Index, Event-Driven Index, Fixed Income Arbitrage, Fixed Income High Yield, Merger Arbitrage Index, Relative Value Arbitrage Index) and main market indices (CAC, SX5E, SPX, UKX et DAX) from 31/01/1990 to 30/11/2004, which represents 180 monthly observations.

3.2.2 Extreme Observations

As a first simple and heuristic test, we determine and calculate for each hedge fund and market index extreme observations, defined as equal to the average minus 3 standard deviations, or to the average minus 2 standard deviations[6]. The number of extreme observations is reported in Tables 5.1 and 5.2.

Table 2
Number of extreme observations

Index	Average-3 stdev	Average-2 stdev
Convertible Arbitrage	3	3
Distressed	1	1
Stat Arb	0	0
Equity Hedge	1	1
Equity Market Neutral	0	0
Event Driven	2	2
FI Arbitrage	2	2
FI High Yield	4	4
Merger Arb	3	3
Relative Value	1	1
CAC	1	7
SX5E	1	8
SPX	1	7
UKX	1	8
DAX	1	7

One extreme observation (average - 3 standard deviations) occurs at the same date (31/08/1998) for all hedge fund indices (except for HFR Statistical Arbitrage). It is also observed (average - 2 standard deviations) for equity markets. The synchronicity of extreme observations supports the idea of the existence of a systematic jump common to the hedge funds and to the equity markets.

[6] Assuming a gaussian distribution, 2 and 3 correspond respectively to the 97.72% and 99.87%.

3.2.3 Principal Component Analysis

In a second step, we use a principal component analysis to extract the common factors between equity market indices and hedge funds. The first five components explain more than 80% of the variance of the indices returns.

Table 3
The first five principal components

Eigenvalue	Explained variance	Cumulated variance
λ_1	48.32%	48.32%
λ_2	13.71%	62.03%
λ_3	9.07%	71.10%
λ_4	6.75%	77.85%
λ_5	4.22%	82.07%

We calculate then for each principal component the skewness and the kurtosis of its observations, and we report the results in Table 5.3.

Table 4
Skewness and Kurtosis of the first five principal components

PC	Skewness	Kurtosis
PC_1	-1.34	7.68
PC_2	-0.42	4.51
PC_3	0.18	3.29
PC_4	-1.12	8.94
PC_5	-1.03	7.47

The first component that explains 48.38% of the variance is characterized by a negative skewness equal to -1.34 and a kurtosis equal to 7.68. We display the pattern of the first five principal components in Figure 5.1.

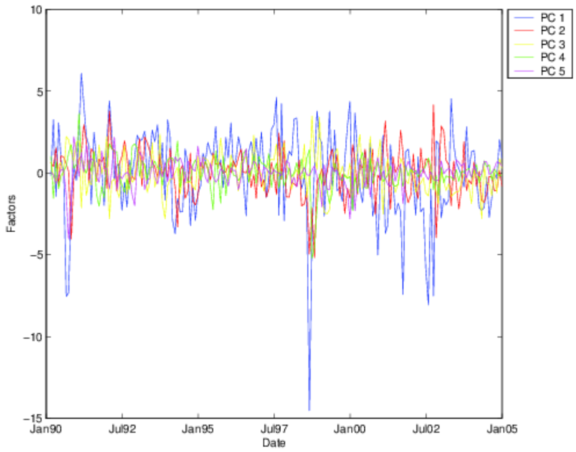


Figure 1
Principal Components

3.2.4 Rolling Correlations with the First Component

We report tables 5.6 , 5.7 and 5.8 yearly rolling correlations of the first component with the different indices. We observe that for hedge fund indices the correlation attains its maximum in 31/01/1999 (which corresponds to the LTCM crisis) for four strategies: Convertible Arbitrage, Distressed, FI High Yield, and Relative Value, in 31/01/2002 for three strategies: Statistical Arbitrage, Equity Hedge, and FI Arbitrage, and in 31/01/2005 for three other hedge fund indices: Equity Market Neutral, Event Driven, and Merger Arbitrage, while it was 31/01/2003 for three equity indices: CAC, SX5E, and SPX.

3.2.5 Non-Parametric Tests for Detecting Jumps

Barndorff-Nielsen and Shephard (2004) provides a convenient non-parametric framework for measuring the relative contribution of jumps to total return variation and for classifying days on which jumps have or have not occurred.

As explained in, e.g., Bollerslev et al. (2008), the realized variance

$$RV_{i,t} = \sum_{j=1}^{M} r_{i,t,j}^2 \tag{4}$$

provides a natural measure of the daily *ex post* variation. In particular, it is well known that for increasingly finer sampling frequencies, or $M \to \infty$, $RV_{i,t}$ consistently estimates the total variation comprised of the integrated variance plus the sum of squared jumps

$$\lim_{M\to\infty} RV_{i,t} = \int_{t-1}^{t} \sigma_i^2(s)ds + \sum_{k=1}^{N_{i,t}} \gamma_{i,t,k}^2 \tag{5}$$

where $N_{i,t}$ denotes the number of within-day jumps on day t, and $\gamma_{i,t,k}$ refer to the size of the k^{th} such jump.

In order to separately measure the two components that make up the total variation in Equation 5.5, Barndorff-Nielsen and Shephard (2004) first proposed the so-called *bipower variation* measure

$$BV_{i,t} = \mu_1^{-2}\left(\frac{M}{M-1}\right)\sum_{j=2}^{M} |r_{i,t,j-1}||r_{i,t,j}| \tag{6}$$

where $\mu_1 = \sqrt{2/\pi} \approx 0.7979$. Under reasonable assumptions, it follows that

$$\lim_{M\to\infty} BV_{i,t} = \int_{t-1}^{t} \sigma_i^2(s)ds \tag{7}$$

and

$$\lim_{M\to\infty} BV_{i,t} = \int_{t-1}^{t} \sigma_i^2(s)ds \tag{8}$$

so that $BV_{i,t}$ consistently estimates the integrated variance for the i^{th} price process, even in the presence of jumps. Thus, as such the contribution to the total variation coming from jumps may be estimated by $RV_{i,t} - BV_{i,t}$, or the relative jump measure $RJ_{i,t} = \frac{RV_{i,t} - BV_{i,t}}{RV_{i,t}}$.

It follows that in the limit as $M \to \infty$, $RJ_t > 0$ only on days for which there are at least one jump, although for finite M sampling variation can occasionally result in $RJ_{i,t} < 0$. For detecting jumps we adopt that ratio which, as shown by Huang and Tauchen (2005), converges to a standard normal distribution after appropriate scaling

$$z_{i,t} = \frac{RJ_{i,t}}{\sqrt{((\frac{2}{\pi})^2 + \pi - 5)\frac{1}{M}\max(1, \frac{TP_{i,t}}{BV_{i,t}^2})}} \tag{9}$$

where $TP_{i,t}$ is the *tripower quarticity* and as shown by Barndorff-Nielsen and Shephard (2004)

$$TP_{i,t} = \mu_{4/3}^{-3} M (\frac{M}{M-2}) \sum_{j=3}^{M} | r_{i,t,j-2} |^{4/3} | r_{i,t,j-1} |^{4/3} | r_{i,t,j} |^{4/3} \to \int_{t-1}^{t} \sigma_i^4(s)ds\sigma_i^2(s)ds \tag{10}$$

with $\mu_{4/3} = 2^{2/3}\Gamma(\frac{7}{6})/\Gamma(\frac{1}{2}) \approx 0.8309$. This statistic exhibits favorable size and power properties, and is quite accurate for univariate jump detection.

The BN-S jump detection scheme being based on asymptotic distribution ($M \to \infty$), a lot of past and on-going research (e.g. Ait-Sahalia et al. (2005), Barndorff-Nielsen and Shephard (2006), and Ait-Sahalia and Yu (2009)) studied the trade-off to be found between the use of all the available data (recorded every second when using intraday data of stocks, exchange rates...) in order to better represent the underlying process, and the necessity of sampling data in order to get rid of market microstructure noise (the optimal sampling frequency adopted in this literature is 5 minutes returns). The BN-S methodology represents the standard approach for non-parametric univariate jump detection on a day-by-day basis. In the sequel, we use that methodology but we have to adapt it to the rare data available on hedge funds. Hence, instead

of using high-frequency data to test for daily jump activity, we change the horizon time and we test for jump activity within one month using daily estimated prices (M=22), and within 1 year using monthly prices (M=12). We start the analysis using the BN-S methodology in order to detect a jump activity both in monthly and yearly data. For that purpose we consider the 2 samples of equity and hedge fund indices described above. Figure 5.2 shows the number of months in which jumps have been detected (respectively for 99% and 95% significance levels[7]). From that figure, we see that equity indices (SX5E, SPX, CAC and DAX) seem to present on average a slightly more important jump activity than hedge fund indices when it is counted within one month: equity indices presented jump activity in 7.75 months on average for 95% significance level (and 2.5 months for 99% significance level), while hedge fund indices jumped in 5.25 months on average for 95% significance level (and 1.75 months for 99% significance level). Figure 5.3 shows the number of years in which jumps have been detected. We see that equity and hedge fund indices present very close jump activity level for the considered period (1990-2004): equity indices presented jump activity in 2.75 years on average for 95% significance level (and 0.5 months for 99% significance level), while hedge fund indices jumped in 2.69 years on average for 95% significance level (and 0.54 years for 99% significance level). The discrepancy between yearly and monthly jump activity for hedge funds (with respect to that of equity indices) may be explained in the latter case by the recourse to daily NAVs which are estimated and subject to the smoothing practice that distorts the statistical properties of returns.

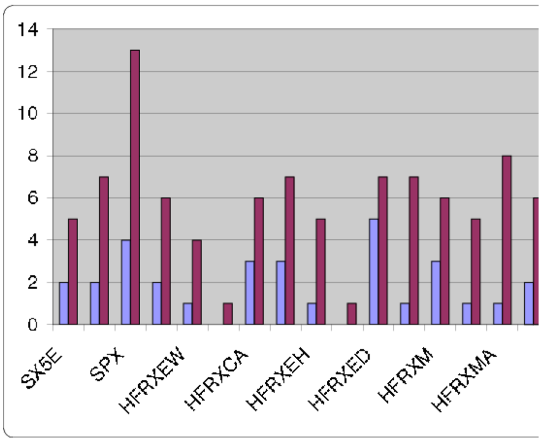


Figure 2
Number of flagged jump months for equity and hedge fund indices from 05/2003 to 09/2007

[7] This choice is less conservative than Bollerslev et al. (2008) who test for the presence of jumps for the 99.9% significance level. This is justified by the more rare data available on hedge funds, while high-frequency data is available for common stocks.

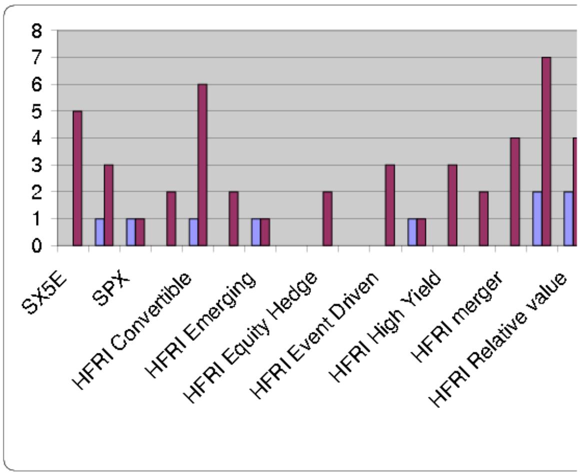


Figure 3

Number of flagged jump years for equity and hedge fund indices from 1991 to 2004

Analyzing index by index, we observe that the different hedge fund indices presented jump activity in common dates with the equity indices, e.g., HFR Equal Weighted, HFR Event Driven, and HFR Merger Arbitrage presented 2 common months (for the 95% significance level) of jump activity with the S&P 500, while HFR Convertible Arbitrage, HFR Equity Hedge presented 3 common months with the same index. We also report common jump activity measured yearly between the DJ EuroStoxx 50 and different hedge fund indices (Convertible, Distressed, Macro, Merger Arbitrage, Mortgage, and Relative Value) during years 1997 and 1998.

In order to further test for the existence of commonality in jumps between equity and hedge fund indices, we calculate correlations between the hedge fund indices z-statistics and the S&P 500 z-statistic. We find that correlations are low and equal on average 0.05. We notice one exception: the HFR Merger Arbitrage whose z-statistic correlation with that of the S&P 500 is equal to 0.31.

Table 5

Correlation between z-statistics of hedge fund indices and the S&P 500 z-statistic

	Correlation
HFRX Equal Weighted Strategies	8.67%
US Convertible Arbitrage Index	10.29%
US Distressed Securities Index	-10.33%
US Equity Hedge	16.04%
US Equity Market Neutral	-21.77%
US Event Driven	31.38%
US Global Hedge Fund	16.08%
US Macro Index	-5.19%
US Merger Arbitrage	10.20%
US Relative Value Arbitrage	-2.39%

The different tests carried out throughout Section 2.2 make acceptable our assumption of the existence of a common jump factor between hedge funds and equity market indices.

4. The CPPI Strategy

Our Financial market is composed of two assets: the riskless asset $(B(t))_{\{t\in[0,T]\}}$ is a deterministic process yielding at each time t an instantaneous interest rate yielding r, and satisfying:

$$dB(t) = rB(t)dt \tag{11}$$

The risky asset is described by the SDE of the hedge fund:

$$dV(t) = V(t^-)[(r+\alpha-c-f(V(t)))dt+\sigma dZ^Q(t)+\gamma dN(t)] \tag{12}$$

In what follows, we recall some known results (see for example Poncet and Portait (1997) and Prigent (2001))

4.1 Risky Exposure

The risky exposure, which we denote $E(t)$, is the amount invested in the hedge fund such that:

$$E(t) = mC(t) \qquad \forall t\in[0,T] \tag{13}$$

where m is a constant called the multiplier, which is predetermined contractually. This parameter is greater than 1, in order to obtain leverage in the risky investment, and make profit of the hedge fund NAV’s increase.

4.2 Wealth

The wealth consists in investing the risky exposure amount $E(t)$ in the risky asset, which is the hedge fund in our case, while investing the balance in the riskless bond $B(t)$. The wealth evolves then following:

$$dW(t) = (W(t)-E(t))\frac{dB(t)}{B(t)}+E(t)\frac{dV(t)}{V(t)} \tag{14}$$

4.3 Floor

The floor is equal to the lowest acceptable value of the portfolio. For simplicity reasons, we will assume that the floor is not stochastic, but evolves as the riskless asset according to:

$$dF(t) = rF(t)dt \tag{15}$$

4.4 Cushion

The cushion is defined as the excess of the wealth's value over the floor's one. Then, the investor is able to determine the amount allocated to the risky asset which is equal to the current value of the cushion multiplied by the multiplier.

When we assume a jump-diffusion process for the risky asset, the cushion evolves following:

$$\frac{dC(t)}{C(t)} = [(1-m)r + m(r+\alpha - c - f(V(t)))]dt + m\sigma dZ^{Q}(t) + m\gamma dN(t) \tag{16}$$

Proof.

$$dC(t) = dW(t) - dF(t) = (W(t) - E(t))\frac{dB(t)}{B(t)} + E(t)\frac{dV(t)}{V(t)} - dF(t)$$

$$= (W(t) - mC(t))\frac{dB(t)}{B(t)} + mC(t)\frac{dV(t)}{V(t)} - dF(t) = (C(t) + F(t) - mC(t))\frac{dB(t)}{B(t)} + mC(t)\frac{dV(t)}{V(t)} - dF(t)$$

but we have $\frac{dB(t)}{B(t)} = \frac{dF(t)}{F(t)}$, thus the result.

4.5 Value of the CPPI

The value of the CPPI payoff at T is equal to:

$$CPPI(T) = Max\left(\frac{W(T)}{W(0)} - K, 0\right)1_{\{min_{0<t<T} C(t) > 0\}} + Max\left(\frac{F(T)}{W(0)} - K, 0\right)1_{\{min_{0<t<T} C(t) < 0\}} \tag{17}$$

5. Modeling the Gap Risk

One issue that should be handled seriously when evaluating CPPI contracts is the valuation of the Gap risk. This risk materializes when, between two respective reallocation dates, the wealth drops below the current floor level, or equivalently, when the Cushion value becomes negative.

In order to assess accurately the price of the CPPI, one needs to price the gap option, which is equal to:

$$\sum_{i=0}^{n-1} \mathbb{E}\left[\left(\frac{V(t_{i+1}) - V(t_i)}{V(t_i)} < -1/m\right) 1_{\cap_{j<i}\left(\frac{m-1}{m} < \frac{V(t_{j+1})}{V(t_j)}\right)}\right]$$

The hedging of the gap risk corresponds to the sum of contingent forward start put options:

$$\sum_{i=0}^{n-1} \mathrm{E}[(\frac{m-1}{m} - \frac{V(t_{i+1})}{V(t_i)})^+ 1_{\cap_{j<i}(\frac{m-1}{m} < \frac{V(t_{j+1})}{V(t_j)})}]$$

Proof.

$$W(t+1) = m(W(t) - F(t))\frac{V(t+1)}{V(t)} + (W(t) - mC(t))\frac{F(t+1)}{F(t)} < F(t+1)$$

$$\frac{V(t+1)}{V(t)} < \frac{F(t+1) - (W(t) - mC(t))\frac{F(t+1)}{F(t)}}{m(W(t) - F(t))}$$

$$\frac{V(t+1)}{V(t)} < \frac{F(t)F(t+1) - (W(t) - mW(t) + mF(t))F(t+1)}{mF(t)(W(t) - F(t))}$$

$$\frac{V(t+1)}{V(t)} < \frac{F(t+1)(F(t) - W(t) + mW(t) - mF(t))}{mF(t)(W(t) - F(t))}$$

$$\frac{V(t+1)}{V(t)} < (m-1)\frac{F(t+1)}{mF(t)}$$

If we approximate $\frac{F(t+1)}{F(t)}$ by 1: $\frac{F(t+1)}{F(t)} \approx 1$, the gap risk between two respective dates materializes if:

$$\frac{V(t+1)}{V(t)} - 1 < -\frac{1}{m}$$

Hence, the *gap option* over a set of n rebalancing dates $\{t_1, t_2, ..., t_i, ..., t_n\}$ will be equal to:

$$\sum_{i=0}^{n-1} \mathrm{E}[(\frac{m-1}{m} - \frac{V(t_{i+1})}{V(t_i)})^+ 1_{\cap_{j<i}(\frac{m-1}{m} < \frac{V(t_{j+1})}{V(t_j)})}]$$

More simply the gap option is equal to $1_{\{min_{0<t<T} C(t)<0\}}$ or $1_{\{min_{t\in\{t(1),t(2),...,t(i),...,t(n)\}} C(t)<0\}}$

6. Numerical Results

For the purpose of our simulations, we choose the following parameters values: V(0)=100, $r_f = 1\%$, multiplier=4, rebalancing=weekly, maturity=1.0 year, $r = 3\%$, $\sigma = 15\%$, $\lambda = 0.5$, $Jump_{mean} = -0.1$, $Jump_{\text{standarddeviation}} = 0.15$ and $N_{MonteCarlo} = 50000$.

Table 5.8 shows as expected that the value of the gap option increases dramatically for high values of the multiplier m. Table 5.9 shows that the value of the gap option increases with the volatility of the hedge fund.

Our results shows that the value of the gap option increases with the volatility of the hedge fund, the standard deviation of the jump, the intensity of the jump, and decreases with the jump mean, which have the opposite effect on the CPPI value. The gap option increases also with the frequency of the rebalancing, as the NAV's values are observed discretely.

Table 6

Yearly rolling correlations (1990/2004)

Date	CAC	SX5E	SPX	UKX	DAX
31/01/1991	73.23%	89.34%	86.41%	72.52%	86.71%
31/01/1992	82.48%	87.23%	54.11%	88.11%	70.90%
31/01/1993	67.01%	78.04%	47.57%	47.63%	56.81%
31/01/1994	80.34%	80.78%	70.13%	74.64%	61.66%
31/01/1995	64.96%	74.64%	88.39%	88.06%	59.45%
31/01/1996	45.79%	84.68%	68.94%	52.90%	72.01%
31/01/1997	80.96%	86.86%	84.72%	54.45%	82.16%
31/01/1998	79.49%	86.31%	84.68%	77.40%	80.83%
31/01/1999	84.20%	88.73%	86.60%	88.69%	89.68%
31/01/2000	85.49%	86.69%	84.00%	61.93%	88.14%
31/01/2001	79.84%	73.56%	69.63%	57.27%	75.94%
31/01/2002	91.58%	91.30%	88.27%	94.13%	93.57%
31/01/2003	93.10%	92.33%	92.48%	86.74%	90.57%
31/01/2004	91.33%	90.05%	91.66%	69.98%	89.66%
31/12/2004	69.73%	82.96%	86.68%	32.93%	70.14%

Table 7
Yearly rolling correlations (1990/2004) (continued)

Date	Convertible	Distressed	Stat Arb	Equity H.	Equity M. N.
31/01/1991	47.36%	77.75%	64.02%	62.88%	-33.77%
31/01/1992	34.03%	73.67%	51.89%	65.98%	-1.15%
31/01/1993	-23.26%	74.55%	65.30%	72.41%	58.93%
31/01/1994	19.68%	80.32%	54.44%	68.10%	50.01%
31/01/1995	50.07%	75.06%	69.43%	60.97%	1.53%
31/01/1996	50.08%	68.58%	51.77%	45.51%	33.78%
31/01/1997	85.43%	66.16%	-16.50%	84.42%	-54.80%
31/01/1998	34.77%	78.53%	66.74%	73.13%	68.38%
31/01/1999	90.41%	96.25%	62.70%	88.06%	60.78%
31/01/2000	36.22%	72.66%	25.53%	92.24%	45.98%
31/01/2001	74.59%	91.81%	31.85%	88.08%	14.34%
31/01/2002	-5.60%	20.41%	85.40%	95.40%	-73.51%
31/01/2003	41.94%	54.91%	80.39%	92.43%	-7.37%
31/01/2004	22.70%	78.01%	25.73%	74.76%	43.16%
31/12/2004	26.69%	90.53%	70.37%	91.85%	74.77%

Table 8
Yearly rolling correlations (1990/2004) (continued)

Date	Event Driven	FI Arb.	FI High Yield	Merger Arb	Relative V.
31/01/1991	89.08%	20.90%	84.25%	78.40%	70.33%
31/01/1992	93.13%	63.35%	72.63%	47.97%	69.01%
31/01/1993	83.27%	33.54%	41.97%	25.41%	23.09%
31/01/1994	85.46%	-11.13%	51.02%	31.42%	39.89%
31/01/1995	60.67%	-54.60%	27.67%	23.68%	45.38%
31/01/1996	70.29%	9.34%	59.95%	15.06%	50.95%
31/01/1997	85.60%	-29.27%	62.92%	44.06%	65.58%
31/01/1998	77.28%	31.06%	72.73%	53.57%	54.60%
31/01/1999	97.83%	25.12%	90.91%	87.84%	91.95%
31/01/2000	82.53%	26.03%	69.73%	11.56%	56.27%
31/01/2001	83.61%	16.11%	63.35%	13.24%	72.16%
31/01/2002	88.52%	74.78%	67.26%	69.94%	36.91%
31/01/2003	87.22%	-74.22%	60.35%	83.89%	67.97%
31/01/2004	80.42%	-4.69%	56.01%	74.00%	59.00%
31/12/2004	97.93%	27.85%	7.75%	91.40%	53.50%

Table 9
The value of the CPPI with multiplier

Multiplier	CPPI	Gap Option
1	104.00	0.61
2	104.00	0.61
3	104.11	0.67
4	104.15	0.91
5	103.13	1.31
6	101.52	1.69
7	100.59	1.71
8	100.00	5.21

Table 10
The value of the CPPI with volatility

Volatility	CPPI	Gap Option
5%	103.99	0.72
10%	104.11	0.82
15%	104.15	0.91
20%	103.86	1.00
25%	103.34	1.11
30%	102.75	1.23

Table 11
The value of the CPPI with Jump mean

Jump Mean	CPPI	Gap Option
-3%	103.84	0.55
-5%	103.92	0.63
-10%	104.15	0.92
-15%	104.42	1.35
-20%	104.73	1.97

Table 12
The value of the CPPI with Jump standard deviation

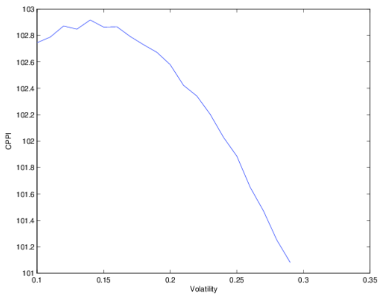
Jump Standard Deviation	CPPI	Gap Option
5%	103.90	0.32
10%	104.03	0.52
15%	104.15	0.91
20%	104.20	1.45

Table 13
The value of the CPPI with λ

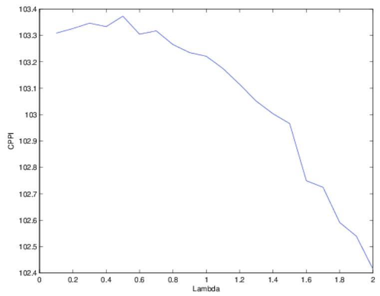
λ	CPPI	Gap Option
0.1	103.84	0.35
0.2	103.94	0.49
0.3	104.02	0.63
0.4	104.10	0.76
0.5	104.15	0.91
1.0	104.39	1.55

Table 14
The value of the CPPI with Frequency

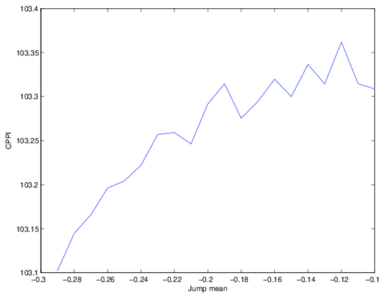
Frequency	CPPI	Gap Option
Daily	104.15	0.76
Weekly	104.15	0.91
Monthly	104.35	1.24



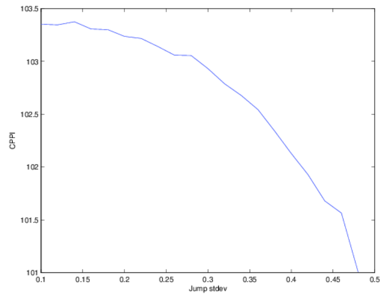
[Volatility]



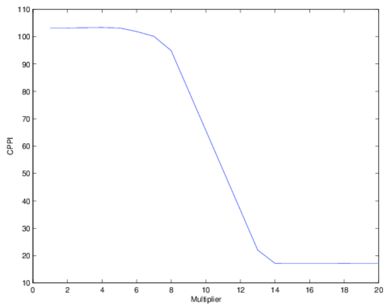
[Lambda]



[Jump mean]

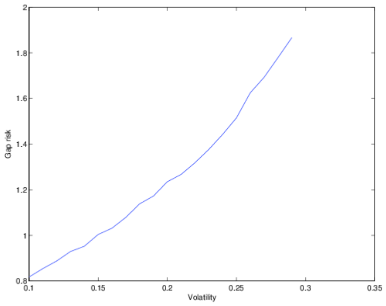


[Jump stdev]

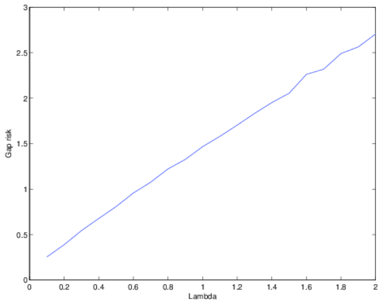


[Multiplier]

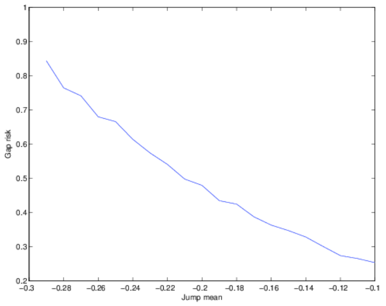
Figure 4
CPPI Sensitivities



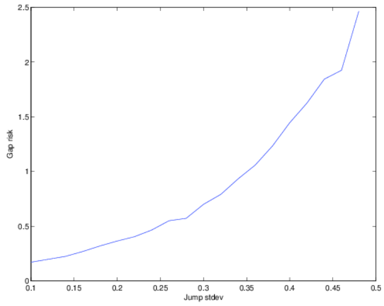
[Volatility]



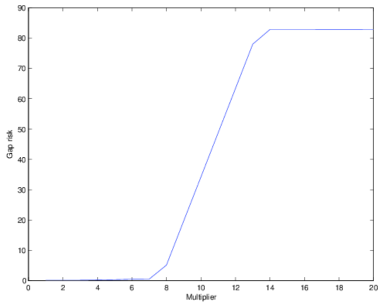
[Lambda]



[Jump mean]



[Jump stdev]



[Multiplier]

Figure 5
Gap Option Sensitivities

7. Conclusion

In order to price CPPI products linked to hedge funds properly we have modeled the hedge fund process through a jump-diffusion one. While Prigent and Tahar (2005), Cont and Tankov (2007) already considered CPPI strategies in presence of jumps in the underlying. Our contribution in this paper was to apply the same analysis to underlyings such as hedge funds, and to investigate the "hedge point of view", more tricky when it comes to hedge funds. For that purpose, we have estimated the parameters of Merton's model from daily time series of hedge fund indices, and we have shown that the jumps in hedge funds are likely to have a common factor with stock markets. This finding is very important as it suggests that *macro-hedge* strategies using equity index options could allow to partly hedge a portfolio of products linked to hedge funds. Then, we exhibited a simple formula for the gap risk. Numerical simulations underline the importance of introducing a jump component for the pricing of the gap option, and the importance of hedging its systematic part in order to reduce this risk.

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Appendix

Jump Process

In this appendix, we start by introducing respectively univariate, multivariate and marked point processes. We then give an example of marked point processes, the compound Poisson process, and we present a simple algorithm for the simulation of a jump-diffusion process.[8]

Univariate Point Process (Poisson Jump Processes)

A point process can be represented as a sequence $\{T_n\}_{n\geq 0}$ of non-negative random variables

$$0 = T_0 < T_1 < T_2 < ... < T_n < \lim_n T_n = +\infty$$

The process can equivalently be represented by its associated counting process N_t :

$$N_t = \sum_{n\in \mathbb{N}^*} 1_{\{T_n \leq t\}}$$

where N_t is called a Poisson point process if it satisfies the three following conditions:

- $N_0 = 0$
- N_t is a process with independent increments
- $N_t - N_s$ is a Poisson random variable with a given parameter $\Lambda_{s,t} = \int_s^t \lambda_u du$, where λ_t

is the intensity of the Poisson point process N_t .

At every date t>0, N_t has the Poisson distribution with parameter λt, that is, it is integer-valued and

$$P[N_t = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

The characteristic function of N_t is given by

$$E[e^{iuN_t}] = exp\{\lambda t(e^{iu} - 1)\}$$

8 See Runggaldier (2003) for a comprehensive presentation on jump processes. See also Tankov and Voltchkova (2009) for a practitioner's guide to jump-diffusion models.

Multivariate Point Process

A K-variate process $N_t = (N_t(1),..., N_t(K))$ is defined by the double sequence $(T_n, Y_n)_{n\geq 1}$ where T_n is a univariate point process and Y_n, $n \geq 1$ a sequence of random variables with values in $\{1,2,...,K\}$, all defined on the same (Ω, F, P). For each $k = 1,..., K$, one may consider the counting process

$$N_t(k) := \sum_{n\geq 1} 1_{\{T_n \leq t\}} 1_{\{Y_n = k\}}$$

Each $N_t(k)$ is a univariate point process and the various $N_t(k)$'s have no common jumps.

Marked Point Process

An E-marked point process is a double sequence $(T_n, Y_n)_{n\geq 1}$ where T_n is a univariate point process corresponding to the n-th occurrence of some phenomenon, and Y_n is a sequence of E-valued random variables corresponding to the mark of this phenomenon. The marked process can equivalently be represented by the counting measure $p(ds, dy)$ where

$$\int_0^t \int_E \gamma(s, y)\, p(ds, dy) = \sum_{n\geq 1} \gamma(T_n, Y_n) 1_{\{T_n \leq t\}} = \sum_{n=1}^{N_t} \gamma(T_n, Y_n)$$

Compound Poisson Process

Let N_t be a Poisson process with parameter λ and $\{Y_i\}_{i\geq 1}$ be a sequence of independent random variables with law f. The process

$$X_t = \sum_{i=1}^{N_t} Y_i$$

is called compound Poisson process. It is characterized by waiting times between jumps $(T_n - T_{n-1})$ that are exponentially distributed, and by random jump sizes.

The compound Poisson process is a particular example of marked and multivariate Poisson processes where $\gamma(t, y) = y$.

Estimation of Jump-Diffusion Processes

Several methods that allow the estimation of the jump-diffusion processes have been developed. The most popular are: Simulated method of moments (Duffie and Singleton (1993)), Simulated maximum likelihood (Durham and Gallant (2002), Brandt and Santa-Clara

(2002)), Markov Chain Monte Carlo and Sequential Bayesian inference (Eraker et al. (1997)), as well as Characteristic function methods (Singleton (2001), Jiang and Knight (2002)). We present this last method below.

The method of empirical characteristic function allows to estimate the model's parameter vector $\theta = (\mu, \sigma, \lambda, b, \delta)$ from historical data and aims at minimizing the integral over u of a weighted distance between the empirical characteristic function and the theoretical characteristic function

$$\int_{\infty}^{\infty} | c_\theta(u) - c(u,n) |^2 \; g(u) du$$

where

$$c(u,n) = \frac{1}{N}\sum_{k=1}^{N} e^{iuX_i}$$

is the empirical characteristic function,

$$c_\theta(u) = exp\{iu(-\frac{1}{2}\sigma^2 - \lambda(e^{\alpha+\delta^2/2} - 1))T - \frac{1}{2}u^2\sigma^2 T + \lambda T(e^{iu\alpha - u^2\delta^2/2} - 1)\}$$

is the characteristic function of the Merton's model.

Simulation of a Jump-Diffusion Process

We retain the first approach of Glasserman (2000) for simulating a jump-diffusion process: We fix the set of dates $0 = t_0 < t_1 < ... < t_n$ without explicitly distinguishing the effects of the jump and diffusion effects.

- generate $Z : N(0,1)$
- generate $N : Poisson(\lambda(t_{i+1} - t_i))$; if $N = 0$, set $M = 0$ and go to Step 4
- generate $\log(Y_1),...,\log(Y_N)$ from their common distribution and set

$M = \log(Y_1) + ... + \log(Y_N)$. In the case of Merton model, $\log Y_j : N(a, b^2)$, and $\sum_{j=1}^{n} \log Y_j : N(an, b^2 n)$, so we have to generate $Z_2 : N(0,1)$ and set $M = aN + b\sqrt{N} Z_2$

- set $X_{t_i+1} = X_{t_i} + (\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} Z + M$

The method relies on two properties of the Poisson process: the increment $N_{t_i+1} - N_{t_i}$ is stationary and has a Poisson distribution $\lambda(t_{i+1} - t_i)$, and is independent of increments of N over $[0, t_i]$.